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Detection of Dirac quanta in Rindler and black hole space–times and the ξ quantisation scheme

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Abstract. A model detector for Dirac quanta based on a simple four-field interaction is analysed, following Unruh's analogous investigation of scalar particle detection in curved space–time. The Dirac ξ scheme in Rindler space is constructed and used to show that an accelerated detector sees an ordinary vacuum as a bath of Dirac quanta with appropriate statistics at a temperature $g/2\pi$. A co-rotating detector constrained near the horizon of the Kerr black hole is similarly shown to detect in the ξ vacuum a fermion bath with temperature proportional to the acceleration. The results provide a physical justification for the ξ quantisation scheme for spin-$\frac{1}{2}$ fields developed and employed in our earlier work.

1. Introduction

The ξ quantisation scheme in a static black-hole space–time refers to a certain definition of positive-frequency modes on the past horizon of the black hole and to a particular choice of the vacuum state of the fields (see §§ 2 and 3). Unruh (1976) showed that the time-dependent collapse problem can be equivalently handled using this scheme leading to an alternative derivation of the well-known Hawking radiation of scalar quanta with a thermal spectrum.

A number of technical arguments, such as the regularity of the derivatives of the Feynman propagator on the future horizon, etc, have been advanced in favour of the ξ scheme over the more conventional η scheme wherein the definition of positive frequency for modes on the horizon is the same as for modes at infinity (Unruh 1976, Fulling 1977). Yet the most pertinent question is whether the ξ definition of a particle at the horizon corresponds at all to a physical particle state. A direct way to settle this question is to investigate the behaviour of model particle detectors in curved space–time (Candelas and Sciama 1977, Hajicek 1977, Meyer 1978, DeWitt 1979), and this analysis for the scalar case (Unruh 1975, 1976) has given a reasonably plausible demonstration that the ξ modes indeed describe physical scalar particles near the horizon of a black hole.

The η and ξ schemes for the Dirac field in a Kerr metric were recently developed by us, using Chandrasekhar's (1976) separable representation, and the ξ scheme was applied to arrive at the Hawking flux of Dirac quanta with appropriate statistics (Iyer and Kumar 1978, 1979a, b). It is therefore natural to pursue the analysis of particle detection for the spin-$\frac{1}{2}$ case also, and investigate the response of a model Dirac detector to different modes of the Dirac field.

In § 2 we first construct the Dirac ξ modes in Rindler space, and then show that an accelerated detector (based on a simple four-field interaction) in a Minkowski vacuum

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sees a bath of Dirac quanta at a temperature equal to $g/2\pi$ (g is the acceleration). In § 3 a co-rotating detector constrained near the horizon of a Kerr black hole (r, θ fixed) is similarly shown to detect in a ξ vacuum a fermion bath at a temperature proportional to its acceleration. The results provide a physical basis for the ξ definition of Dirac quanta in a black-hole space–time.

2. Detection of Dirac quanta in Rindler space

2.1. Dirac ξ scheme in Rindler space-time

The Rindler line element is given by

$$ds^2 = \zeta^2\, d\tau^2 - d\zeta^2 - dx^2 - dy^2; \qquad \zeta \in [0, \infty), \quad \tau \in (-\infty, \infty). \tag{1}$$

The maximal extension of this space is the usual Minkowski space–time, and contains, besides the above exterior region (labelled (+) here), a time-reversed exterior region (−) in addition to the 'interior' future and past regions. Kruskal-like coordinates for the (±) regions are

$$V_\pm = \pm \exp \pm (\tau + \ln \zeta), \tag{2a}$$

$$U_\pm = \mp \exp \mp (\tau - \ln \zeta), \tag{2b}$$

in terms of which the metric is

$$ds^2 = dV\, dU - (dx^2 + dy^2) \tag{3}$$

in all the regions. Note that $U \lessgtr 0$ in (±).

The behaviour of the solution of the Dirac equation in the extended Rindler space–time at the horizon and in the asymptotic limit was obtained in an earlier work (Iyer and Kumar 1977). The normal modes, well-behaved as $\zeta \to \infty$, are given by

$$\psi_{\omega k_1 k_2}(x) = (2\pi)^{-3/2} \exp[i(k_1 x + k_2 y - \omega\tau)](\psi_{\omega k_1 k_2}(\zeta)/\sqrt{\zeta}), \tag{4}$$

where

$$\psi_{\omega k_1 k_2}(\zeta) \xrightarrow[\zeta \to 0]{} \frac{1}{\sqrt{2}}\left[\begin{pmatrix} \chi \\ \sigma^3 \chi \end{pmatrix} \zeta^{i\omega} + A_{\omega k_1 k_2} \begin{pmatrix} \chi \\ -\sigma^3 \chi \end{pmatrix} \zeta^{-i\omega}\right]$$

$$\xrightarrow[\zeta \to \infty]{} \frac{B_{\omega k_1 k_2}}{\sqrt{2}} \begin{pmatrix} \chi \\ (k_1 \sigma^1 + k_2 \sigma^2 - iq\sigma^3)\chi/\mu \end{pmatrix} e^{-q\zeta}, \tag{5}$$

where

$$q = (k_1^2 + k_2^2 + \mu^2)^{1/2}, \qquad \mu = \text{mass of the Dirac particle.}$$

With respect to the positive-definite inner product

$$\langle \psi_1, \psi_2 \rangle = \int (-g)^{1/2} \bar{\psi}_1 \gamma^\tau \psi_2\, d^3x, \tag{6}$$

the modes are orthonormal:

$$\langle \psi_{\omega k_1 k_2}, \psi_{\omega' k_1' k_2'} \rangle = \delta(\omega - \omega')\delta(k_1 - k_1')\delta(k_2 - k_2') \tag{7}$$

(the spin labels in ψ have been suppressed). Similar modes can be defined in the other exterior region. Following the technique employed by Iyer and Kumar (1979a, b), we define a generalised positive-definite inner product in the complete Rindler manifold by

$$(\psi_1, \psi_2) = \langle\psi_{1+}, \psi_{2+}\rangle + \langle\psi_{1-}, \psi_{2-}\rangle, \tag{8}$$

where the $\psi_{1\pm}$ are the restriction of ψ_1 to ($\pm$) regions. The introduction of the generalised inner product makes the treatment of the ξ scheme elegant.

In terms of these modes an arbitrary field may be expanded as

$$\Psi = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_0^{\infty} d\omega\, (a_{\omega k_1 k_2 +}\psi_{\omega k_1 k_2 +} + b^{\dagger}_{\omega k_1 k_2 +}\psi_{-\omega k_1 k_2 +}$$
$$+ a_{\omega k_1 k_2 -}\psi_{-\omega k_1 k_2 -} + b^{\dagger}_{\omega k_1 k_2 -}\psi_{\omega k_1 k_2 -}). \tag{9}$$

Using the canonical field anticommutation relations, we obtain from equation (9)

$$\{a_{\omega k_1 k_2 \pm}, a^{\dagger}_{\omega' k_1' k_2' \pm}\} = \delta(\omega - \omega')\delta(k_1 - k_1')\delta(k_2 - k_2'), \qquad \omega > 0, \tag{10a}$$

$$\{b_{\omega k_1 k_2 \pm}, b^{\dagger}_{\omega' k_1' k_2' \pm}\} = \delta(\omega - \omega')\delta(k_1 - k_1')\delta(k_2 - k_2'), \qquad \omega > 0. \tag{10b}$$

All other anticommutators = 0. The Fulling–Rindler vacuum is given by

$$a_{\omega k_1 k_2 \pm}|0\rangle_\eta = b_{\omega k_1 k_2 \pm}|0\rangle_\eta = 0, \qquad \omega > 0. \tag{11}$$

The positive-frequency definiton for the modes employed above is clearly via the Killing vector $\eta \equiv \partial/\partial\tau$ (i.e. $\exp(-i\omega\tau)$, $\omega > 0$, is a positive frequency mode). This Fulling–Rindler scheme is the customary η scheme of quantisation. The alternative ξ quantisation scheme differs from the above in that here positive frequency is defined via the null Killing vector $\xi \equiv \partial/\partial U$. The ξ definition of positive frequency can be shown to reduce to the standard definition of positive frequency in Minkowski coordinates. As usual, this definition may be equivalently characterised by the analytic behaviour of the mode in the complex U plane. Consider a mode in the extended manifold given by

$$\hat{\psi}^{(1)}_{\omega k_1 k_2} \equiv R(\omega)\psi_{\omega k_1 k_2 +}, \qquad U < 0,$$
$$\equiv R(-\omega)\psi_{\omega k_1 k_2 -}, \qquad U > 0, \tag{12}$$

where

$$R(\omega) = e^{\pi\omega/2}/(2\cosh \pi\omega)^{1/2}.$$

Using the behaviour of $\hat{\psi}_{\omega k_1 k_2 \pm}$ near the past horizon (equation 5) it is seen that $\hat{\psi}^{(1)}_{\omega k_1 k_2}$ is the restriction near real U in the lower half U plane of an analytic and bounded function with a cut along the negative real U axis. Thus $\hat{\psi}^{(1)}_{\omega k_1 k_2}$ is a positive-frequency ξ mode *for all values of* ω. Similarly, the mode

$$\hat{\psi}^{(2)}_{\omega k_1 k_2} \equiv R(\omega)\psi_{-\omega k_1 k_2 +}, \qquad U < 0,$$
$$\equiv -R(-\omega)\psi_{-\omega k_1 k_2 -}, \qquad U > 0, \tag{13}$$

can be shown to be a negative-frequency ξ mode *for all* ω. With respect to the generalised inner product (equation (8)), it is readily established that the ξ modes constructed above are orthonormal:

$$(\hat{\psi}^{(a)}_{\omega k_1 k_2}, \hat{\psi}^{(b)}_{\omega' k_1' k_2'}) = \delta(\omega - \omega')\delta(k_1 - k_1')\delta(k_2 - k_2')\delta^{ab} \qquad (a, b = 1, 2). \tag{14}$$

The ξ expansion of the field is

$$\Psi = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} d\omega\, (\hat{a}_{\omega k_1 k_2}\hat{\psi}^{(1)}_{\omega k_1 k_2} + \hat{b}^{\dagger}_{\omega k_1 k_2}\hat{\psi}^{(2)}_{\omega k_1 k_2}). \tag{15}$$

Inverting the above equation, we obtain the Bogolubov transformation between the η and ξ operators.

$$\hat{a}_{\omega k_1 k_2} = R(\omega) a_{\omega k_1 k_2 +} + R(-\omega) b^{\dagger}_{\omega k_1 k_2 -}, \qquad \omega > 0, \tag{16a}$$

$$\hat{b}^{\dagger}_{\omega k_1 k_2} = R(\omega) b^{\dagger}_{\omega k_1 k_2 +} - R(-\omega) a_{\omega k_1 k_2 -}, \qquad \omega > 0, \tag{16b}$$

$$\hat{a}_{\omega k_1 k_2} = R(\omega) b^{\dagger}_{-\omega k_1 k_2 +} + R(-\omega) a_{-\omega k_1 k_2 -}, \qquad \omega < 0, \tag{16c}$$

$$\hat{b}^{\dagger}_{\omega k_1 k_2} = R(\omega) a_{-\omega k_1 k_2 +} - R(-\omega) b^{\dagger}_{-\omega k_1 k_2 -}, \qquad \omega < 0. \tag{16d}$$

Using equations (10) and (16) we check the consistency of the scheme by verifying that $\hat{a}$ and $\hat{b}$ satisfy the appropriate anticommutation relations,

$$\{\hat{a}_{\omega k_1 k_2}, \hat{a}^{\dagger}_{\omega' k_1' k_2'}\} = \delta(\omega - \omega')\delta(k_1 - k_1')\delta(k_2 - k_2'), \qquad \omega \in (-\infty, \infty), \tag{17a}$$

$$\{\hat{b}_{\omega k_1 k_2}, \hat{b}^{\dagger}_{\omega' k_1' k_2'}\} = \delta(\omega - \omega')\delta(k_1 - k_1')\delta(k_2 - k_2'), \qquad \omega \in (-\infty, \infty). \tag{17b}$$

All other anticommutators vanish.

The Dirac ξ vacuum in Rindler space (Minkowski vacuum) is given by

$$\hat{a}_{\omega k_1 k_2}|0\rangle_{\xi} = \hat{b}_{\omega k_1 k_2}|0\rangle_{\xi} = 0, \qquad \omega \in (-\infty, \infty). \tag{18}$$

2.2. A model detector for Dirac quanta

Our model detector is a box containing a Schrödinger particle localised entirely within the box and admitting of excited energy states. The detection of Dirac quanta is made possible via the simplest four-field interaction with coupling constant ϵ. In the space of the Dirac particle states we write the interaction as

$$\mathscr{H}_{\text{int}}(x) = \epsilon \bar{\Psi}\Psi \phi_{\text{f}}^* \phi_{\text{i}}, \tag{19}$$

where ϕ_{f} amd ϕ_{i} are the Schrödinger wavefunctions of the detector particle and Ψ, $\bar{\Psi}$ are the Dirac field operators. For a uniformly accelerated detector at a fixed $\zeta = \zeta_0$ and $x = y = 0$, the Schrödinger equation for a particle in the box is given by (see e.g. Unruh 1975)

$$\mathrm{i}\frac{\partial \phi}{\partial \tau'} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \zeta^2}\right)\phi + m\frac{(\zeta - \zeta_0)}{\zeta_0}\phi, \tag{20}$$

where τ' is the proper time of the detector: $\tau' = \zeta_0 \tau$. Note that $\zeta_0 = 1/g$ where g is the acceleration of the detector. The detector particle wavefunction may be written as

$$\phi_j = h_j(x, y, \zeta)\exp(-\mathrm{i}E_j \tau'), \tag{21}$$

where h_j are the eigenfunctions of the time-independent Schrödinger equation obtained from equation (20).

The first-order transition probability per unit proper time for excitation from ϕ_{i} to ϕ_{f} is given by

$$\frac{\mathrm{d}P}{\mathrm{d}\tau'} = \lim_{T\to\infty}\frac{|\epsilon|^2}{T}\sum_{|n\rangle}\left|\int_0^T \mathrm{d}\tau' \int_{\text{box}} (-g)^{1/2}\,\mathrm{d}^3x\, \phi_{\text{f}}^*\phi_{\text{i}}\langle n|\bar{\Psi}\Psi|0\rangle\right|^2, \tag{22}$$

where $|n\rangle$ are the possible final states of the Dirac field. Notice that we have chosen the initial state of the field to be a vacuum. Let us suppose this vacuum to be the Fulling–Rindler vacuum defined by equation (11). Using the η expansion of Ψ and $\bar{\Psi}$ (equation (9)), and integrating out the time dependence, one obtains for the detector in the exterior region (+)

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\eta}=2\pi|\epsilon|^2\sum_{\substack{\omega k_1 k_2\\ \omega>0}}\sum_{\substack{\omega' k_1' k_2'\\ \omega'>0}}\delta\left(E_{\mathrm{f}}-E_{\mathrm{i}}+\frac{\omega+\omega'}{\zeta_0}\right)$$
$$\times\left|\int_{\mathrm{box}}(-g)^{1/2}\,\mathrm{d}^3x\,h_{\mathrm{f}}^* h_{\mathrm{i}}\bar{\psi}_{\omega' k_1' k_2' +}(\bar{x})\psi_{-\omega k_1 k_2 +}(\bar{x})\right|^2. \tag{23}$$

Since $E_{\mathrm{f}}>E_{\mathrm{i}}$, the δ function vanishes in the given range of ω, ω' and

$$\mathrm{d}P/\mathrm{d}\tau'|_{\eta}=0. \tag{24}$$

Thus the accelerated detector undergoes no transitions in the Fulling–Rindler vacuum.

On the other hand, if the initial state is the Minkowski or ξ vacuum, the response of the detector is different. Using the η expansion of the field again we obtain

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\xi}=2\pi|\epsilon|^2\sum_{|n\rangle}\sum_{\substack{\omega k_1 k_2\\ \omega>0}}\sum_{\substack{\omega' k_1' k_2'\\ \omega'>0}}\Bigg[\delta\left(E_{\mathrm{f}}-E_{\mathrm{i}}+\frac{\omega'+\omega}{\zeta_0}\right)$$
$$\times\left|\int_{\mathrm{box}}(-g)^{1/2}\,\mathrm{d}^3x\,h_{\mathrm{f}}^* h_{\mathrm{i}}\bar{\psi}_{\omega' k_1' k_2' +}\psi_{\omega k_1 k_2 +}\langle n|a^{\dagger}_{\omega' k_1' k_2' +}a_{\omega k_1 k_2 +}|0\rangle_{\xi}\right|^2$$
$$+\delta\left(E_{\mathrm{f}}-E_{\mathrm{i}}-\frac{\omega'+\omega}{\zeta_0}\right)\left|\int_{\mathrm{box}}(-g)^{1/2}\,\mathrm{d}^3x\,h_{\mathrm{f}}^* h_{\mathrm{i}}\bar{\psi}_{-\omega' k_1' k_2' +}\psi_{\omega k_1 k_2 +}\right.$$
$$\left.\times\langle n|b_{\omega' k_1' k_2' +}a_{\omega k_1 k_2 +}|0\rangle_{\xi}\right|^2+\delta\left(E_{\mathrm{f}}-E_{\mathrm{i}}-\frac{\omega'-\omega}{\zeta_0}\right)$$
$$\times\left|\int_{\mathrm{box}}(-g)^{1/2}\,\mathrm{d}^3x\,h_{\mathrm{f}}^* h_{\mathrm{i}}\bar{\psi}_{-\omega' k_1' k_2' +}\psi_{-\omega k_1 k_2 +}\langle n|b_{\omega' k_1' k_2' +}b^{\dagger}_{\omega k_1 k_2 +}|0\rangle_{\xi}\right|^2\Bigg]. \tag{25}$$

In equations (23) and (25) and below we have converted the integrals into discrete sums for convenience, with appropriate factors of $(\Delta\omega\,\Delta k_1\Delta k_2)^{1/2}$ etc absorbed in $a_{\omega k_1 k_2}$ and $\psi_{\omega k_1 k_2}$. Inverting equation (16) and substituting for a and b in terms of the ξ-operators $\hat{a}$ and $\hat{b}$, the non-vanishing matrix elements in equation (25) are obtained in a straightforward manner. A little manipulation then yields

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\xi}=2\pi|\epsilon|^2\sum_{\substack{k_1 k_2\\ k_1' k_2'}}\sum_{\substack{-\infty\\ \omega,\omega'}}^{\infty}[\exp(2\pi\zeta_0\omega)+1]^{-1}[\exp(2\pi\zeta_0\omega')+1]^{-1}\delta[E_{\mathrm{f}}-E_{\mathrm{i}}-(\omega'+\omega)]$$
$$\times\left|\int_{\mathrm{box}}(-g)^{1/2}\,\mathrm{d}^3x\,h_{\mathrm{f}}^* h_{\mathrm{i}}\bar{\psi}_{-\zeta_0\omega',\,k_1' k_2'}(\bar{x})\psi_{\zeta_0\omega,\,k_1 k_2}(\bar{x})\right|^2. \tag{26}$$

It should be noted that due to the dilation factor ζ_0, $\psi_{\zeta_0\omega,\,k_1 k_2}$ above actually corresponds to a Rindler particle of energy ω with respect to the detector's proper time.

To appreciate the difference in the description of the detection process with respect to the Rindler and Minkowski observers, it is useful to obtain equation (26) alternatively by expanding the field in terms of the ξ modes. This gives

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\xi} = \lim_{T\to\infty} \frac{|\epsilon|^2}{T} \sum_{|n\rangle} \sum_{\substack{k_1k_2\\k_1'k_2'}} \sum_{\substack{-\infty\\ \omega,\omega'}}^{\infty} \times \left| \int_0^T \mathrm{d}\tau' \int_{\text{box}} (-g)^{1/2}\, \mathrm{d}^3x\, \phi_{\mathrm{f}}^* \phi_{\mathrm{i}} \bar{\hat{\psi}}^{(1)}_{\omega' k_1' k_2'} \hat{\psi}^{(2)}_{\omega k_1 k_2} \langle n | \hat{a}^{\dagger}_{\omega' k_1' k_2'} \hat{b}^{\dagger}_{\omega k_1 k_2} |0\rangle_{\xi} \right|^2. \quad (27)$$

Using equations (12) and (13), and performing the τ' integration, we obtain equation (26) again.

From equation (27), the possible final state of the field $|n\rangle$ is a particle–antiparticle pair in the ξ sense. (Note that $\hat{a}^{\dagger}$, $\hat{b}^{\dagger}$ are creation operators of a positive-energy particle and antiparticle for all $\omega \in (-\infty, \infty)$, in contrast to the η operators $a^{\dagger}$, $b^{\dagger}$.) Thus the Minkowski observer sees the detection as a transition from the ground to the excited state of the detector accompanied by an *emission* rather than absorption of a ξ particle–antiparticle pair. There is nothing paradoxical about this since the extra energy for such a process can come from the acceleration mechanism of the detector. For a more general and critical discussion of this point see Davies and Fulling (1977).

On the other hand, equation (25) provides us with the Rindler observer's viewpoint for the same result. According to him, the detector goes to the excited state with either an absorption of a Rindler particle–antiparticle pair (second term of equation (25)) or else by an inelastic scattering of a particle or antiparticle (first and third terms of equation (25)). In either case here, there is a net absorption of energy equal to the transition energy.

The absorption probability per unit proper time corresponding to a given pair of η modes $\omega k_1 k_2$, $\omega' k_1' k_2'$ is given directly by equation (28):

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\substack{\omega k_1 k_2\\ \omega' k_1' k_2'}} = \lim_{T\to\infty} \frac{|\epsilon|^2}{T} \sum_{|n\rangle} \left| \int_0^T \mathrm{d}\tau' \int_{\text{box}} (-g)^{1/2}\, \mathrm{d}^3x \phi_{\mathrm{f}}^* \phi_{\mathrm{i}\eta} \langle 0 | \bar{\Psi}\Psi | \omega k_1 k_2, \omega' k_1' k_2' \rangle_{\eta} \right|^2$$

$$= 2\pi|\epsilon|^2 \delta\left(E_{\mathrm{f}} - E_{\mathrm{i}} - \frac{\omega' + \omega}{\zeta_0}\right) \left| \int_{\text{box}} (-g)^{1/2}\, \mathrm{d}^3x\, h_{\mathrm{f}}^* h_{\mathrm{i}} \bar{\psi}_{-\omega' k_1' k_2'}(\bar{x}) \psi_{\omega k_1 k_2}(\bar{x}) \right|^2. \quad (28)$$

Similarly, the inelastic transition probability for a particle (antiparticle) is also given by equation (28) for $\omega > 0$ ($\omega < 0$), $\omega' < 0$ ($\omega' > 0$). Equation (26) may then be expressed as

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\xi} = \sum_{\substack{k_1k_2\\k_1'k_2'}} \sum_{\substack{-\infty\\ \omega,\omega'}}^{\infty} [\exp(2\pi\zeta_0\omega) + 1]^{-1} [\exp(2\pi\zeta_0\omega') + 1]^{-1} \left.\frac{\mathrm{d}P}{\mathrm{d}\tau'}\right|_{\substack{\zeta_0\omega k_1 k_2\\ \zeta_0\omega' k_1' k_2'}}. \quad (29)$$

The appearance of the product of two statistical distribution factors in the above equation, and the competing process of inelastic scattering in addition to pair absorption are features characteristic of the detection of spin-$\frac{1}{2}$ quanta. The reason for this difference from the scalar case is obvious: a scalar quantum can be singly absorbed (of course its inelastic scattering may occur in higher orders), whereas a spin-$\frac{1}{2}$ detector, even in the lowest order, must either inelastically scatter an electron or else absorb an electron–positron pair. Equation (29) thus clearly establishes the result that the model detector accelerated in Minkowski vacuum 'sees' a thermal bath of Rindler spin-$\frac{1}{2}$

particles and antiparticles with an appropriate statistical distribution corresponding to a temperature equal to $(2\pi\zeta_0)^{-1} = g/2\pi$. Stated differently, the 'vacuum' state for the detector (η vacuum) does not correspond to the physical ξ vacuum of the Minkowski space–time.

3. Detector for Dirac quanta in Kerr space–time

The question that we ask here is: what is the physical vacuum of a Dirac field in a black-hole space–time? Phrased differently, what is the definition of a positive-frequency mode (particle)? In the asymptotically flat region far away from the horizon this definition must obviously coincide with the usual Minkowski definition. There is therefore no ambiguity and here the η and ξ schemes agree. At the horizon, however, the η and ξ definitions of positive frequency differ: the former is defined in terms of $\eta \equiv \partial/\partial t$ and the latter in terms of $\xi \equiv \partial/\partial U$, where U is the Kruskal-like coordinate for the Kerr metric.

Our analysis here shows that the Dirac ξ vacuum, and not the η vacuum in the Kerr metric, is a proper candidate for the physical vacuum state at the past horizon. This conclusion follows naturally since it turns out that *to this ξ vacuum an accelerated co-rotating detector at the past horizon responds in the same manner as a Rindler detector* (§ 2) *responds to the physical Minkowski vacuum.*

Consider a detector at fixed (r, θ) near the horizon of a Kerr black hole, $r = r_+$, and co-rotating with it. By making the transformation

$$\phi' = \phi - \Omega t, \tag{30}$$

where

$$\Omega = (2Mra/|\zeta|^2)[r^2 + a^2 + (2Mra^2 \sin^2\theta/|\zeta|^2)]^{-1},$$

the Kerr metric in the standard Boyer–Lindquist coordinates (t, r, θ, ϕ) can be brought to the form

$$\begin{aligned} ds^2 = {} & \Delta[r^2 + a^2 + (2Mra^2 \sin^2\theta/|\zeta|^2)]^{-1}\, dt^2 - (|\zeta|^2/\Delta)\, dr^2 - |\zeta|^2\, d\theta^2 \\ & - \sin^2\theta[r^2 + a^2 + (2Mra^2 \sin^2\theta/|\zeta|^2)]\, d\phi'^2, \\ & |\zeta|^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2Mr. \end{aligned} \tag{31}$$

Near the horizon

$$\Omega \to \Omega_h = a/(r_+^2 + a^2); \tag{32}$$

Ω_h is the frequency of drag of inertial frames at the horizon. Introduce a coordinate z defined by

$$dz/dr = (|\zeta|^2/\Delta)^{1/2}, \qquad r \to r_+, \quad z \to 0. \tag{33}$$

The metric in a small neighbourhood of the horizon ($z = 0$) reduces to the Rindler-like form

$$\begin{aligned} ds^2 = {} & (1 + 2gz + O(z^2))\, dt'^2 - dz^2 - |\zeta|^2\, d\theta^2 \\ & -\sin^2\theta[r^2 + a^2 + (2Mra^2 \sin^2\theta/|\zeta|^2)]\, d\phi'^2, \end{aligned} \tag{34}$$

where g, the acceleration required to keep the detector fixed at (r, θ), is given by

$$g = (r - M)(\Delta|\zeta|^2)^{-1/2}, \tag{35}$$

and t' is related to the coordinate time t by

$$t' = \Delta^{1/2}[r^2 + a^2 + (2Mra^2 \sin^2 \theta/|\zeta|^2)]^{-1/2} t. \tag{36a}$$

Near the horizon, the proper time for the co-rotating observer becomes

$$t' = \kappa_+ t/g, \tag{36b}$$

where κ_+ is the surface gravity of the Kerr black hole:

$$\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}, \qquad r_\pm = M \pm (M^2 - a^2)^{1/2}. \tag{37}$$

To calculate the probability of excitation of the detector, we need the η and ξ expansions of the Dirac field in the Kerr metric. This is given in Iyer and Kumar (1978, 1979a, b) and we omit the details. Using the notation employed therein, we write the η expansion of the field as

$$\Psi = \sum_{m\lambda\epsilon} \int_{\kappa=+1} \mathrm{d}\omega (a_+(\omega m\lambda\epsilon)\psi_+(\omega m\lambda\epsilon; x) + b_+^\dagger(\omega m\lambda\epsilon)\psi_+(-\omega - m\lambda\epsilon; x)$$
$$+ a_-(\omega m\lambda\epsilon)\psi_-(-\omega - m\lambda\epsilon; x) + b_-^\dagger(\omega m\lambda\epsilon)\psi_-(\omega m\lambda\epsilon; x)), \tag{38}$$

where the $\psi_\pm(\pm\omega, \pm m\lambda\epsilon; x)$ are an orthonormal set of positive frequency η modes, and

$$\kappa = \begin{array}{ll} +1 & \text{if } \epsilon = \mathrm{I},\ \omega > \mu, \\ & \epsilon = \mathrm{II},\ \tilde{\omega} > 0,\ |\omega| > \mu, \\ -1 & \text{if } \epsilon = \mathrm{I},\ \omega < -\mu, \\ & \epsilon = \mathrm{II},\ \tilde{\omega} < 0,\ |\omega| > \mu, \end{array} \tag{39}$$

$$\tilde{\omega} = \omega - m\Omega_\mathrm{h}.$$

If the Dirac field is initially in the η vacuum state, employing the above expansion in equation (22), the first-order excitation probability per unit proper time for a corotating detector near the past horizon is given by

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t'}\right|_\eta = 2\pi|\epsilon|^2 \sum_{\substack{\omega m\lambda \\ \kappa=+1}} \sum_{\substack{\omega' m'\lambda' \\ \kappa'=+1}} \delta\left[E_\mathrm{f} - E_\mathrm{i} + \frac{g}{\kappa_+}(\tilde{\omega} + \tilde{\omega}')\right]$$
$$\times \left| \int_\mathrm{box} (-g)^{1/2}\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi'\, h_\mathrm{f}^* h_\mathrm{i} \bar{\psi}_{\omega' m'\lambda' \mathrm{II}}(\bar{x}) \psi_{-\omega - m\lambda \mathrm{II}}(\bar{x}) \right|^2, \tag{40}$$

which vanishes because, for type II modes, $\tilde{\omega}, \tilde{\omega}' > 0$ in the given range. Thus

$$\mathrm{d}P/\mathrm{d}t'|_\eta = 0. \tag{41}$$

This proves that the co-rotating, accelerated observer sees no 'particles' in the η vacuum. Let us next study how the detector responds to the Dirac ξ vacuum at the past horizon. The ξ expansion of the Dirac field in the Kerr metric is

$$\Psi = \sum_{m\lambda} \int_{\kappa=+1} \mathrm{d}\omega \, (a_+(\omega m\lambda \mathrm{I})\psi_+(\omega m\lambda \mathrm{I}; x) + b_+^\dagger(\omega m\lambda \mathrm{I})\psi_+(-\omega - m\lambda \mathrm{I}; x)$$
$$+ a_-(\omega m\lambda \mathrm{I})\psi_-(-\omega - m\lambda \mathrm{I}; x) + b_-^\dagger(\omega m\lambda \mathrm{I})\psi_-(\omega m\lambda \mathrm{I}; x)$$
$$+ \sum_{m\lambda} \int_{\kappa=\pm1} \mathrm{d}\omega \, (\hat{a}(\omega m\lambda \mathrm{II})\hat{\psi}_{(1)}(\omega m\lambda \mathrm{II}; x) + \hat{b}^\dagger(\omega m\lambda \mathrm{II})\hat{\psi}_{(2)}(\omega m\lambda \mathrm{II}; x)), \tag{42}$$

which differs from the η expansion in the modes localised on the past horizon (type II). Using equation (42) and the relation between ξ modes $\hat{\psi}_{(1)}$, $\hat{\psi}_{(2)}$ and the type II η-modes in the usual exterior region (+), the transition probability per unit proper time in the ξ vacuum can be obtained. The result after a straightforward calculation is

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t'}\right|_{\xi} = 2\pi|\epsilon|^2 \sum_{\substack{\omega m\lambda \\ \kappa=\pm1}} \sum_{\substack{\omega' m'\lambda' \\ \kappa'=\pm1}} \left[\exp\left(\frac{2\pi}{g}\tilde{\omega}\right)+1\right]^{-1} \left[\exp\left(\frac{2\pi}{g}\tilde{\omega}'\right)+1\right]^{-1} \delta[E_{\mathrm{f}} - E_{\mathrm{i}} - (\tilde{\omega} + \tilde{\omega}')]$$
$$\times \left| \int_{\mathrm{box}} (-g)^{1/2} \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi' \, h_{\mathrm{f}}^* h_{\mathrm{i}} \bar{\psi}_{(-\kappa_+\omega'/g)(-\kappa_+ m'/g)\lambda' \mathrm{II}}(\bar{x}) \right.$$
$$\left. \times \psi_{(\kappa_+\omega/g)(\kappa_+ m/g)\lambda \mathrm{II}}(\bar{x}) \right|^2. \tag{43}$$

Notice that only the type II modes contribute at the past horizon since the type I modes are localised at past infinity. It is important to note that $\psi_{(\kappa_+\omega/g)(\kappa_+ m/g)\lambda \mathrm{II}}$ represents a particle of energy $\tilde{\omega}$ with respect to the co-rotating detector's proper time. Next the absorption (or inelastic scattering) probability per unit proper time for a given pair of type II η-modes can be obtained directly from equation (22):

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t'}\right|_{\substack{\omega' m'\lambda' \mathrm{II} \\ \omega m\lambda \mathrm{II}}} = 2\pi|\epsilon|^2 \delta\left[E_{\mathrm{f}} - E_{\mathrm{i}} - \frac{g}{\kappa_+}(\tilde{\omega} + \tilde{\omega}')\right]$$
$$\times \left| \int_{\mathrm{box}} (-g)^{1/2} \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi' \, h_{\mathrm{f}}^* h_{\mathrm{i}} \bar{\psi}_{-\omega' - m'\lambda' \mathrm{II}}(\bar{x}) \psi_{\omega m\lambda \mathrm{II}}(\bar{x}) \right|^2, \tag{44}$$

so that equation (43) can be rewritten as

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t'}\right|_{\xi} = \sum_{\substack{\omega m\lambda \\ \kappa=\pm1}} \sum_{\substack{\omega' m'\lambda' \\ \kappa'=\pm1}} \left[\exp\left(\frac{2\pi}{g}\tilde{\omega}\right)+1\right]^{-1} \left[\exp\left(\frac{2\pi}{g}\tilde{\omega}'\right)+1\right]^{-1} \left.\frac{\mathrm{d}P}{\mathrm{d}t'}\right|_{\substack{(\kappa_+\omega'/g)(\kappa_+ m'/g)\lambda' \mathrm{II} \\ (\kappa_+\omega/g)(k_+ m/g)\lambda \mathrm{II}}}. \tag{45}$$

Equation (45) gives the important result that the co-rotating accelerated detector near the past horizon of a Kerr black hole sees the ξ vacuum state as a fermion bath of η particles and antiparticles at temperature $g/2\pi$. This is exactly analogous to the situation of an accelerated detector in the physical vacuum of the flat space–time discussed in § 2. The ξ vacuum is therefore a natural candidate for the physical vacuum near the past horizon of a black hole. In other words, the definition of the 'ξ particle' at the past horizon corresponds most plausibly to a physical particle state.

4. Conclusion

Drawing upon the analogy with the accelerated detectors in flat space–time we have given a plausible physical justification for the ξ definition of positive frequency for a Dirac field near the past horizon of a Kerr black hole. This strengthens the physical basis of the Unruh–Starobinsky and Hawking effect for Dirac quanta in Kerr space–time derived in the framework of the ξ scheme in our earlier work.

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